# The linearization method and new classes of exact solutions in cosmology

A.V. Yurov[*] and A.V. Astashenok[†]

*I. Kant Russian State University, Theoretical Physics Department, Al.Nevsky St. 14, Kaliningrad 236041, Russia*

We develop a method for constructing exact cosmological solutions of the Einstein equations based on representing them as a second-order linear differential equation. In particular, the method allows using an arbitrary known solution to construct a more general solution parameterized by a set of $3N$ constants, where $N$ is an arbitrary natural number. The large number of free parameters may prove useful for constructing a theoretical model that agrees satisfactorily with the results of astronomical observations. Cosmological solutions on the Randall-Sundrum brane have similar properties. We show that three-parameter solutions in the general case already exhibit inflationary regimes. In contrast to previously studied two-parameter solutions, these three-parameter solutions can describe an exit from inflation without a fine tuning of the parameters and also several consecutive inflationary regimes.

## I. INTRODUCTION

A method for constructing and analyzing exact cosmological solutions of the Einstein equations based on representing them as a second-order linear equation (we call this the linearization method in what follows) was presented in [1] (other methods for constructing exact solutions in cosmology can be found in [2]-[7]). Indeed, it is easy to see that in the case of the of the flat Friedmann metric, the third power of the scale factor $\psi = a^3$ satisfies the equation:

$$\frac{d^2\psi}{dt^2} = \frac{9}{2}\left(\rho - p\right)\psi, \tag{1}$$

where $\rho$ is the density and $p$ is the pressure of the matter filling the universe. Here and hereafter, we use the system of units with $8\pi G/3 = c = 1$. In the case where a minimally coupled scalar field $\phi$ with the self-interaction potential $V(\phi)$ is dominant and in the presence of a cosmological constant with the density $\Lambda$ Eq. (1) formally coincides with the Schrödinger equation

$$\frac{d^2\psi}{dt^2} = (U - \lambda)\psi, \tag{2}$$

where the potential is $U(t) = 9V$, and the spectral parameter is $\lambda = -9\Lambda$. In (2), the quantity $V$ is assumed to be a function of time: $V(t) = V(\phi(t))$, which was called the history of the potential in [1]. Giving an explicit form $U(t)$ together with the corresponding boundary conditions allows finding the general solution of (2). An important consequence of this investigation is that the regime is independent of or weakly dependent on the type of the potential, which is quite significant for the whole theory. Unfortunately, the problem of the end of inflation turns out to be substantially more difficult, and solving it in the framework of the approach described above apparently involves additional assumptions (the authors of [1] proposed modifying the potentials to make them depend on the temperature. The establishment of a Friedmann regime can then be described as a phase transition in the matter of the early universe). The study of Eq. (2) in its application to cosmology was continued in [8] [9], where the Darboux transformation was used to construct new exact solutions (a similar technique was used in [10] to construct exact solutions on the brane and on the encompassing space carrying an orbifold structure).

Here, we present a modification of this method. The Einstein equations imply that the function $\psi_n = a^n$, where $n$ n is an arbitrary (not necessarily integer) number, satisfies a Schrödinger equation with a function $U_n$, that is a linear combination of the density and pressure. If we assume that the universe is filled with a minimally coupled scalar field, then $U_n$ is a linear combination of the potential $V(\phi)$ and the kinetic term, and it is therefore no longer reasonable to call $U_n$ the history of the potential. Hereafter, we call the quantity $U_n$, the "potential" in quotation marks to distinguish it from the self-interaction potential $V$.

---

[*]Electronic address: artyom˙yurov@mail.ru
[†]Electronic address: artyom.art@gmail.com

Fixing a "potential" we can find solutions for the function $\psi_n$ and thus find the scale factor $a_n = \psi_n^{1/n}$ as a function of time. In general, a solution of the Schrödinger equation has the form

$$\psi_n(t) = c_1\psi_1(t) + c_2\psi_2(t), \tag{3}$$

i.e., it depends on two arbitrary constants. The scale factor in turn depends on three parameters: $a = a(n, c_1, c_2; t)$. This circumstance allows constructing multiparameter solutions of the Einstein equations as follows. We assume that the quantity $a(t)$ is determined, for example, by astronomical observation. Raising this function to the $n_1$th power, we obtain a function denoted by $\psi_1$ in (3). In the next step, we find $\psi_2$ based on the condition of linear independence, which we write as

$$\psi_1\frac{d\psi_2}{dt} - \psi_2\frac{d\psi_2}{dt} = 1,$$

after which we find a three-parameter solution $a = a(n_1, c_1^{(1)}, c_2^{(1)}; t)$. We can now repeat this procedure and find a six-parameter solution

$$a = a(n_1, c_1^{(1)}, c_2^{(1)}; n_2, c_1^{(2)}, c_2^{(2)}; t)$$

and so on. After $N$ steps, we obtain a solution that depends on $3N$ parameters. On the other hand, knowing the scale factor in the Friedmann cosmology allows computing all other characteristics, such as the Hubble parameter, the acceleration, the density, and the pressure. The photometric distance, which is a crucial quantity for testing models, is a function of six parameters; therefore, using the above procedure sufficiently many times, we can make the model consistent with observations.

Unfortunately, the effectiveness of this method is considerably reduced when considering Friedmann models with a nonzero spatial curvature. In this case, the function $\psi_n = a^n$ for an arbitrary $n$ n satisfies not the Schrödinger equation but an equation with an additional nonlinear term. There are no meaningful techniques for integrating such an equation; it is therefore generally difficult to find a solution depending on two arbitrary constants, which is necessary for a complete examination of the problem, similar to the investigation in[1]. It would be interesting to show that the conclusions in [1] also hold for $k = \pm 1$. Fortunately, solving the complicated nonlinear equation discussed above is unnecessary for this: it suffices to set $n = 1$ instead of $n = 3$. In this case, the wave function is the scale factor itself. One of the two cosmological Einstein equations in the Friedmann metric is a second-order linear equation with a potential proportional (with a minus sign) to $\rho + 3p/c^2$. The physical meaning of the "potential" is also more transparent. If we set the cosmological term to zero, then if the potential is negative, then the strong energy condition is satisfied, and if it is positive, then this condition is violated, which generally implies inflation. Therefore, the problem of studying inflationary regimes becomes much simpler, even when compared with flat ($k = 0$) models.

This paper is organized as follows. In Sec. 2, we formulate the linearization method exactly, i.e., the reduction of the Friedmann equations to a Schrödinger equation for arbitrary $n$. Moreover, we prove a similar assertion for the cosmology on the RandallSundrum I brane (RS-I). In Sec. 3, we give several examples of exact solutions. We see that three-parameter families of solutions are much richer in properties than the two-parameter solutions studied in [1]. In particular, such "potentials" can lead to solutions describing several inflationary stages. Perhaps our universe is currently undergoing one of them.

## II. THE LINEARIZATION METHOD

We consider the Einstein equations in the Friedmann metric:

$$\frac{\dot{a}^2}{a^2} = \rho - \frac{k}{a^2}, \tag{4}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{2}\left(\rho + 3p\right). \tag{5}$$

We assume that $a = a(t)$, $p = p(t)$, $\rho = \rho(t)$ is a solution of these equations for $k = 0$. Then the function $\psi_n = a^n$ is a solution of the Schrödinger equation

$$\ddot{\psi}_n = U_n(t)\psi_n, \tag{6}$$

where the "potential" is

$$U_n(t) = n^2\rho - \frac{3n}{2}(\rho + p). \tag{7}$$

If the universe is filled with a scalar field $\phi$ with the Lagrangian $L = \frac{\dot{\phi}^2}{2} - V(\phi)$, then

$$U_n = \frac{n(n-3)}{2}\dot{\phi}^2 + n^2 V(\phi). \tag{8}$$

**Remark 1**. For $n = 3$, the "potential" is $U_3 = 9V(\phi(t))$. This case was studied in detail in [1], where this quantity, as already noted, was called the history of the potential because $U_3$ appears in the equations as a function of time ($U = U(t)$), not a function of the field variable $\phi$. Nevertheless, the potential $U \sim V$, the physical meaning of the potential $U$ therefore seems clear. If $n \neq 3$, then $U_n$ is a certain linear combination of the kinetic term $\dot{\phi}^2/2$ and the self-interaction potential $V$ (. (7), (8)), and the physical meaning of such a $U$ is not so evident.

Nevertheless, the effectiveness of the method presented in [1] (and developed in [8], [9]), precisely consists in reducing a complex nonlinear problem to a linear equation. This allows nding full two-parameter solutions, which exhibit inflationary behavior under very general assumptions. The fact that $U$ basically coincides with $V$ was not used anywhere in those papers and therefore played no role. Similarly, we here consider a generalization of this method to arbitrary $n$.

Furthermore, the physical meaning of the "potential" $U_3$ is clear only for a universe filled with a scalar field. If we consider a universe in which, for example, electromagnetic radiation is dominant, the physical meaning of the quantity $U_3$ becomes ambiguous.

**Remark 2**. If we assume that the universe contains a nonzero vacuum energy with the density $\rho_\Lambda c^2$, in addition to the matter fields, then Eq. (6) takes the form of the spectral problem

$$\ddot{\psi}_n = (U_n(t) - \lambda_n)\psi_n, \tag{9}$$

where the spectral parameter is $\lambda_n = -3n^2\rho_\Lambda/3$. Just as for Eq. (2), we can consider a problem for the eigenvalues and the eigenfunctions of Eq. (9) if we specify homogeneous initial conditions. As noted in [1], Eq. (2) has the form of a quantum mechanical problem with a discrete spectrum. The fact that each such solution only admits a bounded or countable set of allowed values of the cosmological constant (if we specify homogeneous initial conditions) may clarify the question of the actual value of the cosmological constant.

We note that if a solution of (9) is known, then we can use (4) and (9) to find the scalar field and the potential:

$$\phi(t) = \pm\frac{\sqrt{2}}{\sqrt{3n}}\int dt\sqrt{\frac{\dot{\psi}_n^2}{\psi_n^2} - U_n + \lambda_n}, \tag{10}$$

$$V(t) = \frac{1}{3}\left(\frac{U_n}{n} + \frac{3-n}{n^2}\left(\frac{\dot{\psi}_n^2}{\psi_n^2} + \lambda_n\right)\right). \tag{11}$$

We can obtain the dependence $V = V(\phi)$ from these expressions, although it is clearly not always possible to do this explicitly. In the general case, a solution of (6) has the form

$$\Psi_n = c_1\psi_n + c_2\hat{\psi}_n, \tag{12}$$

where $\hat{\psi}_n$ is a linearly independent solution with the same potential:

$$\hat{\psi}_n(t) = \psi_n(t)\int^t \frac{dt'}{\psi_n^2(t')} \equiv \psi_n(t)\xi(t). \tag{13}$$

Equation (12) allows proving the following assertion.
**Assertion**. Let $a = a(t)$ be a solution of (4) and (5) for $k = 0$ and the corresponding $\rho$ and $p$. Then the three-parameter function $a_n = a(t; c_1, c_2, n)$ of the form

$$a_n = a\left(c_1 + c_2\int\frac{dt}{a^{2n}}\right)^{1/n}, \tag{14}$$

is a solution of (4 and 5) for a new energy density $\rho_n$ and pressure $p_n$ satisfying the condition

$$n^2\rho_n - \frac{3n}{2}\left(\rho_n + p_n/c^2\right) = n^2\rho - \frac{3n}{2}\left(\rho + p/c^2\right). \tag{15}$$

**Remark 3**. In general, this assertion holds for $k = 0$. If $k = \pm 1$, then it holds only if $n = 0, 1$.

**Remark 4**. A similar assertion can be formulated for solutions describing the RS-I brane. In this case, the Friedmann system is modified by taking the brane tension $\sigma$ into account and becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \rho\left(1 + \frac{\rho}{2\sigma}\right),$$

$$-2\frac{\ddot{a}}{a} = \rho + 3p + \frac{\rho}{\sigma}(2\rho + 3p),$$

(16)

It is easy to see that the function $\psi_n \equiv a^n$ satisfies the linear Schrödinger equation

$$\frac{\ddot{\psi}_n}{\psi_n} = W_n,$$

(17)

with the potential

$$W_n = \frac{n}{2}\left(2n\rho - 3(\rho + p) + \frac{\rho}{\lambda}(n\rho - 3(\rho + p))\right) =$$

$$= \frac{n}{2}\left[2n(K + V) - 6K + \frac{1}{\lambda}(K + V)(n(K + V) - 6K)\right],$$

(18)

where $V = V(\phi)$ and $K = \dot{\phi}^2/2$. Hence, the method of generating $3N$-parameter families of solutions described above can also be applied to cosmology on a brane. An example of an exact two-parameter solution (with $n = 3$) was given in [11], where a linearization method for a simple anisotropic cosmological model was described.

In what follows, we consider several examples of exact solutions based on an integrable potential of the Schrödinger operator, following [1].

## III. GENERATING EXACT SOLUTIONS WITH A GIVEN $U_n(t)$

We consider the model "potentials"

$$U_n(t) = \mu^2 t^2,$$

(A)

$$U_n(t) = -\frac{2\lambda_0}{\cosh^2(\lambda_0 t)}.$$

(B)

We study the solutions for potentials (A) and (B) for the possible existence of inflationary regimes and exit from inflation.

**Potential (A)**. For potential (A), the solution of Eq. (9) with zero boundary conditions as $t \to \pm\infty$ has the form

$$\psi_n = A^n H_s(\mu t)\exp(-\mu t^2/2),$$

where $A$ is a constant and $H_s(\mu t)$ are the Hermite polynomials of order $s$. The corresponding evolution of the scale factor is

$$a(t) = A H_s^{1/n}(\mu t)\exp(-\mu t^2/2n).$$

(19)

In the simplest case, we have $\psi_n = \exp(-\mu t^2/2)$. In this case, the dependen on $n$ can be eliminated by a simple redenition of the parameter $\mu$. This solution was considered in [1], and we therefore do not consider it further.

Choosing the wave function of the ground state as the solution of problem (9) is not obligatory. We can consider solutions in $L^2$ that correspond to excited levels, but these solutions cannot be used on the entire interval on which they are defined. The point is that according to the oscillation theorem, the wave function of the $s$th excited level has $s$ zeros, each of which in the cosmological context corresponds to a singularity with the scale factor tending to zero if $n > 0$, and to innity (a Big Rip singularity) if $n < 0$. We can use solutions for excited levels with numbers $s > 1$ and consider the dynamics described by a part of the eigenfunction on an interval. Such a universe begins and/or ends its existence at the corresponding singularity. For example, if we take the function $\psi_n \sim H_1(\mu t)\exp(-\mu t^2/2)$ as a

solution of (9) with the "potential" $U_n = \mu^2 t^2$ and let $n$ be -1, then the evolution of the scale factor in this universe can be written as

$$a(t) = \frac{a_0 t_0}{t} e^{\mu(t^2 - t_0^2)/2}. \tag{20}$$

This solution describes a universe in which the scale factor is equal to innity at $t = 0$ (a Big Rip singularity), takes its minimum value at $t = \mu^{-1/2}$ and then begins a never-ending inflationary stage. Using formulas (10 and 11), we can also find the asymptotic behavior of the scalar field and the potential for $t \sim 0$ and as $t \to \infty$. The kinetic term of the energy density of the scalar field is negative, i.e., the evolution described by (20) corresponds to a phantom field (see, e.g., [12]-[16] for phantom fields). Equation (20) corresponds to a negative cosmological constant $\Lambda = -3\mu$.

**Potential (B)**. The solution for potential (B) for $\lambda_n = -\lambda^2 \leq 0$ (which corresponds to a nonnegative value of the cosmological constant) has the form

$$\psi_n = c_1(\lambda - \lambda_0 \tanh(\lambda_0 t)) e^{\lambda t} + c_2(\lambda + \lambda_0 \tanh(\lambda_0 t)) e^{-\lambda t}. \tag{21}$$

If $\lambda = \lambda_0$, then formula (21) simplifies considerably:

$$\psi_n = \frac{C}{\cosh(\lambda_0 t)}.$$

For positive $n$, the evolution of the scale factor corresponding to the function $\psi_n$ describes with an inflationary regime on the interval $(-\infty, t_0)$, where $t_0$ is the inflection point of the function $\cosh^{1/n}(\lambda_0 t)$.

An interesting class of solutions can also be obtained in the simplest case $\lambda_n = 0$. Then (21) becomes

$$\psi_n = C \tanh(\lambda_0 t). \tag{22}$$

If $n \geq 1$, then the solution for the scale factor describes a universe leaving the singular state at the moment $t = 0$ and asymptotically approaching a stationary state as $t \to \infty$ with $\ddot{a} < 0$ during the entire evolution. If $0 < n < 1$, then the universe undergoes an inationary phase until a certain instant and then asymptotically approaches a stationary state.

In the case where $\lambda > \lambda_0$, $c_1 > 0$ and $c_2 = 0$, there is a solution

$$\psi_n = c_1(\lambda - \lambda_0 \tanh(\lambda_0 t)) e^{\lambda t},$$

$$a = c_1^{1/n}(\lambda - \lambda_0 \tanh(\lambda_0 t))^{1/n} e^{\lambda t/n}. \tag{23}$$

This solution corresponds to evolution without singularities and is interesting because for certain values of the parameters $\lambda_0$, $\lambda$ and $n$, formulas (23) describe a universe that undergoes inflationary expansion on some interval $(0, t_1)$, then inflation ends, and a secondary inflationary period begins at a time $t_2 > t_1$. To determine when such a situation occurs, we consider $\ddot{a}$ at the initial instant. Using (23), we see that

$$\ddot{a}(0) \sim (1 - n)y^4 - 2y^2 + 1,$$

where we introduce the notation $y = \lambda_0/\lambda < 1$. Therefore, the second derivative of the scale factor is nonnegative at the initial instant if

$$0 < y^2 \leq y_0^2 = \frac{1 - \sqrt{n}}{1 - n}.$$

A further investigation shows that satisfaction of this condition means that the universe immediately enters an inflationary regime if $n \geq 1$. If $y^2 > y_0^2$, then solution (23) describes an evolution with an inflationary regime starting at a time $t_0 > 0$. If $n < 1$ then in the narrow interval $y_0^2 - \Delta < y^2 < y_0^2$, where $\Delta \ll y_0^2$, there is a period of noninflationary expansion, after which the universe again enters a period of accelerating expansion. For example, if $n = 0.25$ and $\lambda_0 = 0.8\lambda$, then the universe undergoes accelerating expansion in the interval $(0, 0.08/\lambda)$, followed by a noninflationary expansion stage of length approximately $0.4/\lambda$. For $t > 5/\lambda$, the scale factor can be considered to change according the exponential law $a \sim e^{4\lambda t}$.

We conclude by considering the asymptotic behavior of the potential of the scalar field during the early stage of the evolution of such a universe. We limit ourself to terms linear in time in the decomposition. Using (10) and (11), we find that for $t \ll 1/\lambda$,

$$\phi(t) \approx \phi_0 \pm \sqrt{\frac{2}{3n}} y^2 \lambda t,$$

$$U(t) \approx \frac{\lambda^2}{3} \left( \frac{3-n}{n^2} y^4 - \frac{6}{n^2} y^2 - \frac{2(3-n)}{n^2} y^4 \left( 1 - y^2 \right) \lambda t \right).$$

Therefore, if there is an initial inflation, then it corresponds to a slow-roll regime. The potential slowly decreases as the scalar eld increases or decreases linearly.

## IV. CONCLUSION

We have proposed a relatively simple method for generating exact solutions of the Einstein-Friedmann equations. We used mathematical transformations to reduce the problem to solving the Schrödinger equation for the function $a^n$ with a "potential" proportional to $n^2\rho - 3n(\rho + p/c^2)/2$. We focused on studying the inflationary regime and the exit from it.

Three-parameter families of solutions exhibit a much richer repertoire of behaviors than do the two-parameter solutions studied in [1]. Nevertheless, these solutions, like the solutions described in [1], have inflationary phases under quite general assumptions. This is an indication that inflation is not something exotic found only in a limited number of models. On the contrary, an inflationary regime seems a fairly common occurrence in cosmology not requiring any special initial assumptions.

Unlike two-parameter solutions, three-parameter solutions can have several consecutive inflationary phases, i.e., they can not only describe inflation but also describe an exit from the inflationary phase without a special fine tuning of the parameters. Moreover, the existence of such solutions can be seen as indirect evidence for the existence of a *unified* realistic model that contains not only an inflationary phase (with an exit from it without a fine tuning) in the early universe stage but also a later inflation, which our own universe is perhaps undergoing currently.

Finally, the procedure for constructing a three-parameter solution can be repeated an arbitrary number $N$ of times, yielding $3N$-parameter solutions. The presence of a sufficiently large number of free parameters can be considered a defect of a theory but, on the contrary, can be useful for fitting the theoretical model to observational data. We plan to return to this question in future investigations. Hence, the described method may prove quite fruitful in cosmology.

[1] V. M. Zhuravlev, S. V. Chervon, and V. K. Shchigolev, JETP, **87**, 223 (1998).

[2] J. D. Barrow, Phys. Rev. D, **49**, 3055 (1994).

[3] R. Maartens, D. R. Taylor, and N. Roussos, Phys. Rev. D, **52**, 3358 (1995).

[4] G. F. R. Ellis and M. S. Madsen, Class. Q. Grav., **8**, 667 (1991).

[5] J. E. Lidsey, Class. Q. Grav., **8**, 923 (1991).

[6] J. D. Barrow and P. Saich, Class. Q. Grav., **10**, 279 (1993).

[7] P. Parson and J. D. Barrow, Class. Q. Grav., **12**, 1715 (1995).

[8] A.V. Yurov, Phantom Scalar Fields Result in Inflation Rather Than Big Rip, astro-ph/0305019.

[9] S. D. Vereshchagin and A. V. Yurov, Theor. Math. Phys., **139**, 787 (2004).

[10] A.V. Yurov, V.A. Yurov, Phys. Rev. D **72**, 026003 (2005).

[11] A.V. Yurov, A.V. Astashenok, V.A. Yurov, Grav. Cosm. **14**, 8 (2008).

[12] Cepa J. Astron. Astrophys. **422**, 831 (2004).

[13] A.A. Andrianov, F. Cannata, A.Y. Kamenshchik, Phys. Rev. D **72**, 043531 (2005).

[14] Yurov A.V., Moruno P.M. and González-Díaz P.F. Nucl. Phys. **B759**, 320 (2006).

[15] E. Elizalde, S. Nojiri, and S. Odintsov, Phys. Rev. D, **70**, 043539 (2004).

[16] S. Nojiri and S. Odintsov, Phys. Lett. B, **562**, 147 (2003).